Chapter 5 Integrals: an informal introduction

The main goal of this chapter is to study **areas of plane regions bounded by curves**. Unlike in the previous chapters, we will take a rather informal approach and leave the conceptual study to the next course MATH1014.

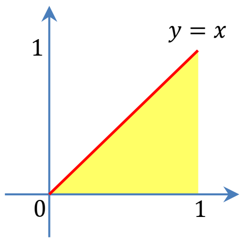
1. Area and integrals

Let's start with a simple case when a **non-negative continuous** function f is given, and we first study the area of the region bounded between the graph of f and the x-axis, on a bounded closed interval $[a, b]$. If the graph of this function f is so nice that this region has an elementary shape (e.g. rectangle, triangle, circular disk, etc.), then the task is easy.

Example 5.1 For each of the following functions $f: [0, 1] \to \mathbb{R}$, find the area of the region bounded between the graph of f and the x-axis on $[0, 1]$.

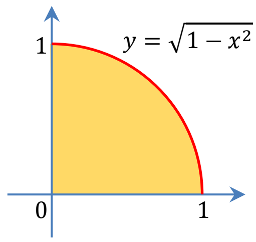
(a) $f(x) = x$ (b) $f(x) = \sqrt{1 - x^2}$

Solution:



(a) The region is a right-angled triangle whose height and base width are both 1 unit. Therefore the required area is

$$A = \frac{1}{2}(1)(1) = \frac{1}{2}.$$



(b) The region is a quarter of a circular disk of radius 1 unit centered at the origin. Therefore the required area is

$$A = \frac{1}{4} \cdot \pi(1)^2 = \frac{\pi}{4}.$$

Definition 5.2 Let $a < b$ and let $f: [a, b] \to [0, +\infty)$ be a **non-negative** continuous function.

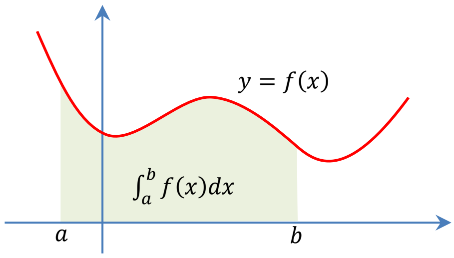
⊙ The ***integral of f over the interval $[a, b]$***, denoted by

$$\int_a^b f(x)dx,$$

is the area of the region in the coordinate plane bounded by the graph of f, the x-axis, the vertical line $x = a$ and the vertical line $x = b$.

⊙ The process of finding the integral of a function is called ***integration***. So to ***integrate*** the function f means to find its integral, i.e. to compute the area of the above region.

⊙ Given an integral as above, the function f is called the ***integrand***, and the numbers a and b are called the ***lower limit*** and the ***upper limit*** of the integral respectively.



Example 5.3 In Example 5.1, we have deduced that

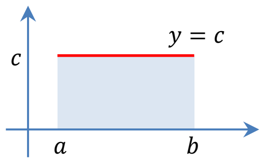
$$\int_0^1 x\,dx = \frac{1}{2} \qquad \text{and} \qquad \int_0^1 \sqrt{1-x^2}\,dx = \frac{\pi}{4},$$

as these integrals represent areas of a triangle and a quarter of a circular disk respectively.

Example 5.4 Let $a < b$ be real numbers and let c be a positive real number. If f is the constant function $f(x) = c$, then

$$\int_a^b f(x)dx = c(b-a), \qquad \text{or simply} \quad \int_a^b c\,dx = c(b-a),$$

as it represents the area of a rectangle with base width $(b-a)$ units and height c units.



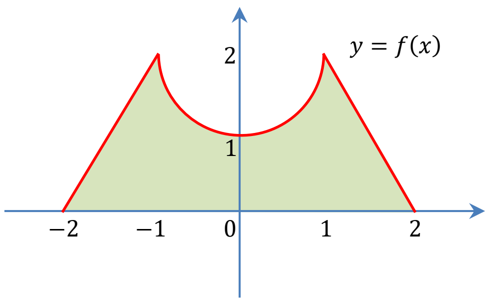
Example 5.5 Let $f: [-2, 2] \to [0, +\infty)$ be the function defined by

$$f(x) = \begin{cases} 4+2x & \text{if } x \le -1 \\ 2-\sqrt{1-x^2} & \text{if } -1 < x < 1 \\ 4-2x & \text{if } x \ge 1 \end{cases}.$$

Evaluate the integral

$$\int_{-2}^{2} f(x)dx.$$

Solution:
The graph of f is sketched in the diagram on the right. The required integral represents the area of a trapezium having parallel edges with lengths 2 and 4 and having height 2, with a semicircular disk with radius 1 removed. So



$$\int_{-2}^{2} f(x)dx = \frac{1}{2}(2+4)\cdot 2 - \frac{1}{2}\pi(1)^2$$
$$= 6 - \frac{\pi}{2}.$$

Remark 5.6 Some remarks about the notation of an integral:

(i) The symbol dx in the integral is called a ***differential*** (cf. Remark 4.24), and its precise meaning is again well out of the scope of this course. Here we understand dx just as a symbol which indicates that the horizontal axis is labeled as the "x"-axis.

(ii) The integral $\int_a^b f(x)dx$ is <u>just a number</u>, and its value does not depend on the symbol x. In fact, we can replace x by any other symbol without changing the value of the integral:

$$\int_a^b f(x)dx = \int_a^b f(u)du = \int_a^b f(t)dt\,.$$

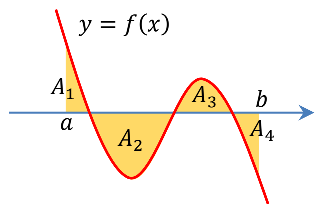
We say that the symbol x in the integral $\int_a^b f(x)dx$ is a ***dummy variable***, which means that the integral actually <u>contains no such variable</u>. To avoid confusion, the dummy variable should not also appear in the upper or lower limits of the integral, or outside the integral:

$$\int_a^x f(x)dx, \qquad \int_a^b g(a)da, \qquad \frac{1}{t}\int_a^b h(t)dt \qquad \text{are not allowed.}$$

When f is also allowed to take negative values, we extend the definition of its integral as follows.

Definition 5.7 Let $a < b$ be real numbers and let $f\colon [a,b] \to \mathbb{R}$ be a continuous function. Then $\int_a^b f(x)dx$ is defined to be the "**net area**" under the graph of f on $[a,b]$. In other words,

$$\int_a^b f(x)dx \coloneqq \text{(the area above the } x\text{-axis and below the graph of } f)$$
$$- \text{(the area below the } x\text{-axis and above the graph of } f).$$



$\int_a^b f(x)\,dx = A_1 - A_2 + A_3 - A_4$

Example 5.8 Let $a < b$ and c be real numbers (now c is allowed to be negative). Similar to Example 5.4 we still have

$$\int_a^b c\,dx = c(b-a).$$

Example 5.9 Let $f\colon [-4,4] \to \mathbb{R}$ be the function

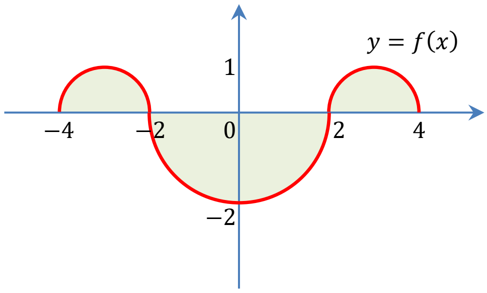
$$f(x) = \begin{cases} \sqrt{1-(|x|-3)^2} & \text{if } |x| \geq 2 \\ -\sqrt{4-x^2} & \text{if } |x| < 2 \end{cases}.$$

Evaluate

$$\int_{-4}^4 f(x)dx.$$

Solution:

The graph of f is sketched in the diagram on the right. The required integral represents the area of two semicircular disks both with radius 1 minus the area of a semicircular disk with radius 2. So



$$\int_{-4}^{4} f(x)dx = 2\left(\frac{1}{2}\pi(1)^2\right) - \frac{1}{2}\pi(2)^2 = -\pi.$$

The following are some **properties of integrals**.

Theorem 5.10 (Area property) *Let a, b, c be real numbers and f be a continuous function. Then*

$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx.$$

Remark 5.11 Let a and b be real numbers and f be a function. To make sense of the above theorem, we use the following **convention** in case the lower limit a is greater than or equal to the upper limit b:

$$\int_a^b f(x)dx \coloneqq -\int_b^a f(x)dx.$$

From this convention we immediately obtain

$$\int_a^a f(x)dx = 0.$$

Example 5.12 Let f be a continuous function such that

$$\int_{-1}^{3} f(x)dx = -2 \qquad \text{and} \qquad \int_{-1}^{6} f(x)dx = 4.$$

Compute the following integrals.

(a) $\displaystyle\int_2^2 f(x)dx$ (b) $\displaystyle\int_3^6 f(y)dy$

Solution:

(a) $\displaystyle\int_2^2 f(x)dx = 0$

(b)

$$\int_3^6 f(y)dy = \int_3^6 f(x)dx = \int_3^{-1} f(x)dx + \int_{-1}^{6} f(x)dx = -\int_{-1}^{3} f(x)dx + \int_{-1}^{6} f(x)dx$$
$$= -(-2) + 4 = 6$$

Theorem 5.13 (Integral inequality) *Let $a < b$ be real numbers and f, g be continuous functions. If $f(x) \leq g(x)$ for every $x \in [a, b]$, then*

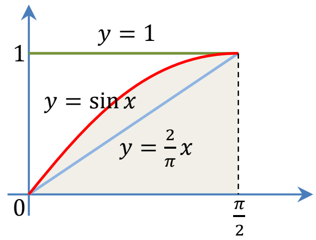
$$\int_a^b f(x)dx \leq \int_a^b g(x)dx.$$

Example 5.14 The integral

$$\int_0^{\pi/2} \sin x \, dx$$

represents the area of a region that does not take an elementary shape. Let's try to **estimate** the value of this integral. Let $f, g, h: [0, \pi/2] \to \mathbb{R}$ be the continuous functions

$$f(x) = \frac{2}{\pi}x, \qquad g(x) = \sin x \qquad \text{and} \qquad h(x) = 1.$$



Note that the graph of f is the line segment joining the two points $(0, 0)$ and $\left(\frac{\pi}{2}, 1\right)$, and the graph of h is a horizontal line segment.

⊙ It is obvious that $g(x) = \sin x \leq 1 = h(x)$ for all $x \in \left[0, \frac{\pi}{2}\right]$. On the other hand, since

$$g''(x) = -\sin x < 0 \qquad \text{for every } x \in \left(0, \frac{\pi}{2}\right),$$

we know that the graph of g is **concave downward** on the interval $(0, \pi/2)$; therefore the **line segment** joining the points $(0, 0)$ and $(\pi/2, 1)$ (i.e. the graph of f) must lie **below** the graph of g (cf. Remark 4.75). Combining, we have

$$f(x) \leq g(x) \leq h(x) \qquad \text{for every } x \in \left[0, \frac{\pi}{2}\right].$$

Thus

$$\int_0^{\pi/2} f(x)dx \leq \int_0^{\pi/2} g(x)dx \leq \int_0^{\pi/2} h(x)dx.$$

⊙ The region below the graph of f on $\left[0, \frac{\pi}{2}\right]$ is a right-angled triangle, whose area is

$$\int_0^{\pi/2} f(x)dx = \frac{1}{2}\left(\frac{\pi}{2}\right)(1) = \frac{\pi}{4}.$$

⊙ The region below the graph of h on $\left[0, \frac{\pi}{2}\right]$ is a rectangle, whose area is

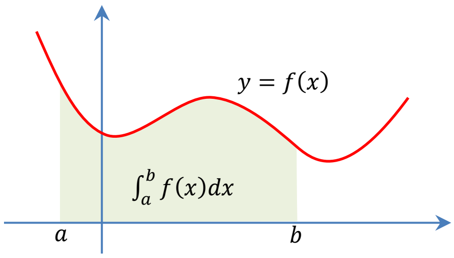
$$\int_0^{\pi/2} h(x)dx = \left(\frac{\pi}{2}\right)(1) = \frac{\pi}{2}.$$

Therefore we deduce that

$$\frac{\pi}{4} \leq \int_0^{\pi/2} \sin x \, dx \leq \frac{\pi}{2}.$$

2. Antiderivatives

In the previous section, we have seen some examples of calculating the integral of a continuous function f, if the region bounded between the graph of f and the x-axis **has an elementary shape** (e.g. rectangle, triangle, circular disk, etc.). However, the graphs of most continuous functions do not have such nice elementary shapes.



Fortunately, there is a nice (and unexpected) connection of the integral with **derivatives**, which helps us accurately compute integrals of a large class of continuous functions. This connection is called the **Newton-Leibniz formula**, or also known as (the second version of) the **Fundamental Theorem of Calculus**. The proof of this important theorem, together with a more comprehensive and conceptual study of integrals, will be done in MATH1014.

Theorem 5.15 (Newton-Leibniz) *Let* $f: [a, b] \to \mathbb{R}$ *and* $F: [a, b] \to \mathbb{R}$ *be continuous functions. If* F *is differentiable on* (a, b) *and* $F' = f$ *on* (a, b)*, then*

$$\int_a^b f(x)dx = F(b) - F(a).$$

Example 5.16 With the Newton-Leibniz formula, we can now **accurately calculate** the integral

$$\int_0^{\pi/2} \sin x \, dx$$

which we have tried to estimate in Example 5.14. Let $f(x) = \sin x$ and $F(x) = -\cos x$. Then f and F are both continuous on $\mathbb{R}$. Since F is differentiable on $\mathbb{R}$ and $F' = f$ on $(0, \pi/2)$, by the Newton-Leibniz formula we have

$$\begin{aligned} \int_0^{\pi/2} \sin x \, dx &= \int_0^{\pi/2} f(x)dx = F\left(\frac{\pi}{2}\right) - F(0) \\ &= \left(-\cos\frac{\pi}{2}\right) - (-\cos 0) = 0 - (-1) \\ &= 1. \end{aligned}$$

Note that this accurate result agrees with our previous estimation that

$$\frac{\pi}{4} \le \int_0^{\pi/2} \sin x \, dx \le \frac{\pi}{2}.$$

The Newton-Leibniz formula suggests that in order to calculate the integral of a given continuous function f, we need to **find a function F whose derivative is f**, i.e. $F' = f$. In the following, let's study systematically how to find such a function F.

Definition 5.17 Let f be a function. If F is a differentiable function such that $F' = f$ on an open interval (a, b), then F is called an ***antiderivative*** of f ***on*** $(\boldsymbol{a}, \boldsymbol{b})$.

Example 5.18 Let $f: \mathbb{R} \to \mathbb{R}$ be the function

$$f(x) = \cos x\,.$$

Then the function

$$F(x) = \sin x$$

is an antiderivative of f on $\mathbb{R}$. The function

$$G(x) = \sin x + 1$$

is also another antiderivative of f on $\mathbb{R}$.

Remark 5.19 Let f be a function defined on an open interval (a, b), and suppose that F is an antiderivative of f on (a, b). Then

- for each constant function C, $F + C$ is also an antiderivative of f on (a, b);
- according to Corollary 4.64 (which is a consequence of the Mean Value Theorem), **every** antiderivative of f on (a, b) **must be** of the form $F + C$ where C is a constant function.

Therefore whenever we have found an antiderivative of a function on an open interval, to include all the antiderivatives on the interval we just need to add to it an arbitrary constant function C.

Because of the important connection between derivatives and integrals established by the Newton-Leibniz formula, we adopt the following **notation of the antiderivative** instead of the "naturally expected" notation

$$f^{(-1)} \qquad \text{or} \qquad \left(\frac{d}{dx}\right)^{-1} f(x).$$

Some people even call antiderivatives to be ***indefinite integrals*** because of such a connection.

Definition 5.20 Let f be a function defined on an open interval. Then the ***collection of all antiderivatives of f*** is denoted as

$$\int f(x)dx\,.$$

Thus if F is an antiderivative of f, then we write

$$\int f(x)dx = F(x) + C,$$

where C is an arbitrary constant.

Example 5.21 Find the derivative of the function

$$f(x) = x\ln x - x.$$

Hence, find all the antiderivatives of the function

$$g(x) = \ln x$$

on its natural domain $(0, +\infty)$.

Solution:
On the interval $(0, +\infty)$, the derivative of f is given by

$$f'(x) = (1)(\ln x) + (x)\left(\frac{1}{x}\right) - 1 = \ln x = g(x).$$

Since $f' = g$, the antiderivatives of g on $(0, +\infty)$ are all those functions of the form $f + C$, i.e.

$$\int g(x)dx = f(x) + C = x\ln x - x + C,$$

where C is an arbitrary constant function.

Remark 5.22 Unlike in an integral, the x appearing in the antiderivative $\int f(x)dx$ is **NOT a dummy variable** because it indeed represents (a family of) functions that depend on the variable x.

By "reversing" the rules of differentiation, we obtain some **rules of antidifferentiation**. Here are the first two rules, and two other rules will be introduced in the next two sections.

Theorem 5.23 *Let f and g be functions having antiderivatives and c be a real number. Then*

(i) $\displaystyle\int (f \pm g)(x)dx = \int f(x)dx \pm \int g(x)dx$;

(ii) $\displaystyle\int (cf)(x)dx = c\int f(x)dx$.

Proof. These follow immediately from the corresponding rules of differentiation, Theorem 3.21. ∎

Remark 5.24 Note that **there is NO "product rule" or "quotient rule"** for antiderivatives. In particular, please do not apply **INCORRECT** formulas such as

$$\int f(x)g(x)dx = \left(\int f(x)dx\right)g(x) + f(x)\left(\int g(x)dx\right) \qquad \times$$

or

$$\int f(x)g(x)dx = \int f(x)dx \int g(x)dx, \qquad \times$$

which **do not hold** in general! We will come up with some ways to handle antiderivatives of products of two specific functions in the next two sections.

Remark 5.25 From our knowledge of derivatives of elementary functions, we immediately obtain the following **antidifferentiation formulas**, in each of which C is an arbitrary constant.

(i) ***Polynomials and power functions***

- $\displaystyle\int x^r\,dx = \frac{1}{r+1}x^{r+1} + C$ for every $r \neq -1$
- $\displaystyle\int \frac{1}{x}\,dx = \ln|x| + C$

(ii) ***Exponential functions***

- $\displaystyle\int e^x\,dx = e^x + C$
- $\displaystyle\int a^x\,dx = \frac{a^x}{\ln a} + C$ for every $a \in (0,1) \cup (1,+\infty)$

(iii) ***Trigonometric functions***

- $\displaystyle\int \cos x\,dx = \sin x + C$
- $\displaystyle\int \sin x\,dx = -\cos x + C$
- $\displaystyle\int \sec^2 x\,dx = \tan x + C$
- $\displaystyle\int \csc^2 x\,dx = -\cot x + C$
- $\displaystyle\int \sec x \tan x\,dx = \sec x + C$
- $\displaystyle\int \csc x \cot x\,dx = -\csc x + C$

(iv) ***Algebraic functions related to inverse trigonometric functions***

- $\displaystyle\int \frac{1}{\sqrt{1-x^2}}\,dx = \arcsin x + C$
- $\displaystyle\int \frac{1}{1+x^2}\,dx = \arctan x + C$
- $\displaystyle\int \frac{1}{x\sqrt{x^2-1}}\,dx = \operatorname{arcsec}|x| + C$

Example 5.26 Evaluate the following antiderivatives.

(a) $\displaystyle\int e^{t+3}dt$

(b) $\displaystyle\int \frac{(u-1)(u+2)}{\sqrt{u}}du$

(c) $\displaystyle\int \frac{2x^2}{1+x^2}dx$

(d) $\displaystyle\int \tan^2\theta\, d\theta$

Solution:

(a)

$$\int e^{t+3}dt = \int e^t e^3 dt = e^3 \int e^t dt$$
$$= e^3 e^t + C = e^{t+3} + C,$$

where C is an arbitrary constant.

(b) It is hard to find antiderivatives of products of functions, so let's expand it into a sum.

$$\int \frac{(u-1)(u+2)}{\sqrt{u}}du = \int \frac{u^2+u-2}{\sqrt{u}}du$$
$$= \int \left(u^{\frac{3}{2}} + u^{\frac{1}{2}} - 2u^{-\frac{1}{2}}\right)du$$
$$= \int u^{\frac{3}{2}}du + \int u^{\frac{1}{2}}du - 2\int u^{-\frac{1}{2}}du$$
$$= \frac{1}{5/2}u^{\frac{5}{2}} + \frac{1}{3/2}u^{\frac{3}{2}} - 2\left(\frac{1}{1/2}u^{\frac{1}{2}}\right) + C$$
$$= \frac{2}{5}u^{\frac{5}{2}} + \frac{2}{3}u^{\frac{3}{2}} - 4u^{\frac{1}{2}} + C,$$

where C is an arbitrary constant.

(c) Long division gives $\frac{2x^2}{1+x^2} = 2 - \frac{2}{1+x^2}$. Thus

$$\int \frac{2x^2}{1+x^2}dx = \int \left(2 - \frac{2}{1+x^2}\right)dx = 2\int dx - 2\int \frac{1}{1+x^2}dx$$
$$= 2x - 2\arctan x + C,$$

where C is an arbitrary constant.

(d)

$$\int \tan^2\theta\, d\theta = \int (\sec^2\theta - 1)d\theta = \int \sec^2\theta\, d\theta - \int d\theta$$
$$= \tan\theta - \theta + C,$$

where C is an arbitrary constant.

Combining the knowledge of antiderivatives together with the Newton-Leibniz formula

$$\int_a^b f(x)dx = F(b) - F(a),$$

we can now (accurately) **compute integrals using antiderivatives**, even if they represent areas of regions which do not take an elementary shape.

Remark 5.27 The Newton-Leibniz formula says that we can evaluate integrals using **any** antiderivative, so in practice we usually use the **simplest** one by dropping the arbitrary constant C.

Remark 5.28 There are many commonly used notations

$$[F(x)]_a^b, \qquad F(x)|_a^b, \qquad F(x)]_a^b$$

which all mean $F(b) - F(a)$. So the Newton-Leibniz formula can also be written as

$$\int_a^b f(x)dx = [F(x)]_a^b.$$

Example 5.29

⊙ Since $\frac{d}{dx}\left(\frac{1}{3}x^3\right) = x^2$, we have

$$\int_0^1 x^2 dx = \left[\frac{1}{3}x^3\right]_0^1 = \frac{1}{3}(1)^3 - \frac{1}{3}(0)^3 = \frac{1}{3}.$$

⊙ Since $\frac{d}{dx}e^x = e^x$, we have

$$\int_0^a e^x dx = [e^x]_0^a = e^a - e^0 = e^a - 1 \qquad \text{for each } a > 0.$$

⊙ Since $\frac{d}{dx}\ln x = \frac{1}{x}$, we have

$$\int_1^a \frac{1}{x}dx = [\ln x]_1^a = \ln a - \ln 1 = \ln a \qquad \text{for each } a > 1.$$

⊙ (cf. Examples 5.14 and 5.16) Since $\frac{d}{dx}(-\cos x) = \sin x$, we have

$$\int_0^a \sin x\, dx = [-\cos x]_0^a = (-\cos a) - (-\cos 0) = 1 - \cos a \qquad \text{for each } a > 0.$$

⊙ Since $\frac{d}{dx}\arctan x = \frac{1}{1+x^2}$, we have

$$\int_0^1 \frac{1}{1+x^2}dx = [\arctan x]_0^1 = \arctan 1 - \arctan 0 = \frac{\pi}{4}.$$

Thanks again to the Newton-Leibniz formula, we obtain **properties of integrals** that are similar to the corresponding properties of antiderivatives.

Theorem 5.30 *Let a, b, c be real numbers and f, g be continuous functions. Then*

(i) $\displaystyle\int_a^b (f \pm g)(x)dx = \int_a^b f(x)dx \pm \int_a^b g(x)dx,$

(ii) $\displaystyle\int_a^b (cf)(x)dx = c\int_a^b f(x)dx.$

Example 5.31 Let's evaluate the following integrals.

(a)

$$\begin{aligned}\int_1^4 (5x+3)\sqrt{x}\,dx &= \int_1^4 \left(5x^{\frac{3}{2}} + 3x^{\frac{1}{2}}\right)dx = \left[2x^{\frac{5}{2}} + 2x^{\frac{3}{2}}\right]_1^4 \\ &= \left[2(4)^{\frac{5}{2}} + 2(4)^{\frac{3}{2}}\right] - \left[2(1)^{\frac{5}{2}} + 2(1)^{\frac{3}{2}}\right] \\ &= 80 - 4 = 76\end{aligned}$$

(b)

$$\begin{aligned}\int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \left(\frac{1}{\cos^2\theta} + \frac{1}{\sin^2\theta}\right)d\theta &= \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} (\sec^2\theta + \csc^2\theta)d\theta = [\tan\theta - \cot\theta]_{\pi/4}^{\pi/3} \\ &= \left(\tan\frac{\pi}{3} - \cot\frac{\pi}{3}\right) - \left(\tan\frac{\pi}{4} - \cot\frac{\pi}{4}\right) \\ &= \left(\sqrt{3} - \frac{1}{\sqrt{3}}\right) - (1-1) = \frac{2}{\sqrt{3}}\end{aligned}$$

Remark 5.32 We need to be careful about checking the conditions when computing integrals using the Newton-Leibniz formula.

⊙ Although $\frac{d}{dx}\left(-\frac{1}{x}\right) = \frac{1}{x^2}$, it is **<u>INCORRECT</u>** to say that

$$\int_{-1}^1 \frac{1}{x^2}dx = \left[-\frac{1}{x}\right]_{-1}^1 = -2 \qquad \times$$

by the Newton-Leibniz formula. This is incorrect because $-\frac{1}{x}$ is not an antiderivative of $\frac{1}{x^2}$ **on** $\boldsymbol{(-1,1)}$; note that the function $-\frac{1}{x}$ is not differentiable at $0 \in (-1,1)$. In fact, we will learn in MATH1014 that the integral $\int_{-1}^1 \frac{1}{x^2}dx$ **does not exist**.

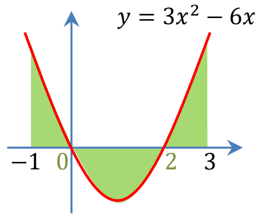
When handling an integrand that involve absolutes values or other **piecewise-defined functions**, we do not want to find its antiderivative directly. Instead, we **break down the interval into pieces** so that the integrand is elementary on each piece, before applying the Newton-Leibniz formula.

Example 5.33 Evaluate the following integrals.

(a) $\displaystyle\int_{-1}^{3} |3x^2 - 6x|dx$ (b) $\displaystyle\int_{0}^{2} e^{|1-x|}dx$ (c) $\displaystyle\int_{0}^{3\pi/2} |\sin x|dx$

Solution:

(a) Since

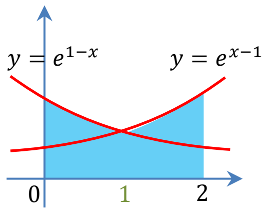


$$3x^2 - 6x = 3x(x-2)\begin{cases} \geq 0 & \text{if } x \geq 2 \text{ or } x \leq 0 \\ < 0 & \text{if } 0 < x < 2 \end{cases},$$

we have

$$\begin{aligned}\int_{-1}^{3} |3x^2 - 6x|dx &= \int_{-1}^{0} (3x^2 - 6x)dx + \int_{0}^{2} -(3x^2 - 6x)dx + \int_{2}^{3} (3x^2 - 6x)dx \\ &= [x^3 - 3x^2]_{-1}^{0} + [-x^3 + 3x^2]_{0}^{2} + [x^3 - 3x^2]_{2}^{3} \\ &= 0 - (-4) + 4 - 0 + 0 - (-4) \\ &= 12.\end{aligned}$$

(b) Since

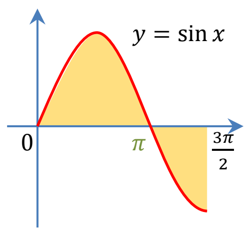


$$1 - x\begin{cases} \geq 0 & \text{if } x \leq 1 \\ < 0 & \text{if } x > 1 \end{cases},$$

we have

$$\begin{aligned}\int_{0}^{2} e^{|1-x|}dx &= \int_{0}^{1} e^{1-x}dx + \int_{1}^{2} e^{x-1}dx \\ &= [-e^{1-x}]_{0}^{1} + [e^{x-1}]_{1}^{2} \\ &= -1 - (-e) + e - 1 \\ &= 2e - 2.\end{aligned}$$

(c) Since



$$\sin x\begin{cases} \geq 0 & \text{if } 0 \leq x \leq \pi \\ < 0 & \text{if } \pi < x \leq 3\pi/2 \end{cases},$$

we have

$$\begin{aligned}\int_{0}^{3\pi/2} |\sin x|dx &= \int_{0}^{\pi} \sin x\, dx + \int_{\pi}^{3\pi/2} -\sin x\, dx \\ &= [-\cos x]_{0}^{\pi} + [\cos x]_{\pi}^{3\pi/2} \\ &= 1 - (-1) + 0 - (-1) \\ &= 3.\end{aligned}$$

Example 5.34 Using the **Newton-Leibniz formula** together with the **integral inequality**, we can establish some inequalities about the cosine and sine functions as follows.

(i) Since $\cos t \leq 1$ for every $t \geq 0$, the integral inequality implies that

$$\int_0^x \cos t \, dt \leq \int_0^x 1 \, dt, \qquad \text{i.e. } \sin x \leq x$$

for every $x \geq 0$.

(ii) Since $\sin t \leq t$ for every $t \geq 0$ by (i), the integral inequality implies that

$$\int_0^x \sin t \, dt \leq \int_0^x t \, dt, \qquad \text{i.e. } 1 - \cos x \leq \frac{1}{2}x^2$$

for every $x \geq 0$.

(iii) Since $\cos t \geq 1 - \frac{1}{2}t^2$ for every $t \geq 0$ by (ii), the integral inequality implies that

$$\int_0^x \cos t \, dt \geq \int_0^x \left(1 - \frac{1}{2}t^2\right) dt, \qquad \text{i.e. } \sin x \geq x - \frac{1}{6}x^3$$

for every $x \geq 0$.

(iv) Since $\sin t \geq t - \frac{1}{6}t^3$ for every $t \geq 0$ by (iii), the integral inequality implies that

$$\int_0^x \sin t \, dt \geq \int_0^x \left(t - \frac{1}{6}t^3\right) dt, \qquad \text{i.e. } 1 - \cos x \geq \frac{1}{2}x^2 - \frac{1}{24}x^4$$

for every $x \geq 0$.

Continuing in the same way, we can conclude that for each non-negative integer n, the inequalities

$$1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \cdots - \frac{1}{(4n+2)!}x^{4n+2} \leq \cos x \leq 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \cdots + \frac{1}{(4n)!}x^{4n}$$

$$x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \cdots - \frac{1}{(4n+3)!}x^{4n+3} \leq \sin x \leq x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \cdots + \frac{1}{(4n+1)!}x^{4n+1}$$

hold for every $x \geq 0$. (One can check that these inequalities hold for $x < 0$ also.)

These inequalities suggest that the cosine and sine functions may be approximated by polynomials

$$\cos x \approx 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \cdots + \frac{(-1)^n}{(2n)!}x^{2n}$$

$$\sin x \approx x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \cdots + \frac{(-1)^n}{(2n+1)!}x^{2n+1}$$

if n is large. In fact, for $n = 3$, the approximation errors on the interval $[0, \pi/4]$ are at most

$$0 \leq \cos x - \left(1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6\right) \leq 3.59 \times 10^{-6},$$

$$0 \leq \sin x - \left(x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7\right) \leq 3.13 \times 10^{-7}.$$

We will learn more about polynomial approximation of functions in MATH1014.

3. Integration by substitution

The following rule of antidifferentiation corresponds to the **chain rule** of differentiation.

Theorem 5.35 (Substitution rule for antiderivatives) *Let g be a differentiable function and f be a function which is well-defined on the range of g. If F is an antiderivative of f, then*

$$\int f(g(x))g'(x)dx = F(g(x)) + C,$$

where C is an arbitrary constant.

Proof. If F is an antiderivative of f, then $F' = f$. So by the chain rule of differentiation,

$$\frac{d}{dx}F(g(x)) = F'(g(x))g'(x) = f(g(x))g'(x)$$

for every x in the domain of g. ∎

Remark 5.36 To remember the substitution rule, we may let $u = g(x)$. Then the **differential** of u is given by $du = g'(x)dx$ (cf. Remark 4.24), so

$$\int f(g(x))g'(x)dx = \int f(u)du = F(u) + C = F(g(x)) + C.$$

Computing differentials is therefore very useful.

Corollary 5.37 *If $\int f(x)dx = F(x) + C$, then for every $a \neq 0$ and every $b \in \mathbb{R}$ we have*

$$\int f(ax+b)dx = \frac{1}{a}F(ax+b) + C,$$

where C is an arbitrary constant.

Proof. Apply the substitution rule with $u = ax + b$. ∎

Example 5.38 Evaluate the following antiderivatives:

(a) $\displaystyle\int \frac{1}{\sqrt{2x-5}}dx$

(b) $\displaystyle\int e^x \cos(e^x)\, dx$

(c) $\displaystyle\int \frac{(\ln x)^2}{x}dx$

Solutions:

(a) Let $u = 2x - 5$. Then $du = 2dx$, so

$$\int \frac{1}{\sqrt{2x-5}}\,dx = \int \frac{1}{\sqrt{u}} \cdot \frac{1}{2}\,du = \frac{1}{2} \cdot 2\sqrt{u} + C = \sqrt{2x-5} + C,$$

where C is an arbitrary constant.

(b) Let $u = e^x$. Then $du = e^x dx$, so

$$\int e^x \cos(e^x)\,dx = \int \cos u\,du = \sin u + C = \sin(e^x) + C,$$

where C is an arbitrary constant.

(c) Let $u = \ln x$. Then $du = (1/x)dx$, so

$$\int \frac{(\ln x)^2}{x}\,dx = \int u^2 du = \frac{1}{3}u^3 + C = \frac{1}{3}(\ln x)^3 + C,$$

where C is an arbitrary constant.

Example 5.39 Evaluate

$$\int \tan x\,dx.$$

Don't just stop at this step! We need to present the answer using the **original variable** x.

Solution:

Let $u = \cos x$. Then $du = -\sin x\,dx$, so by the substitution rule

$$\int \tan x\,dx = \int \frac{\sin x}{\cos x}\,dx = -\int \frac{1}{u}\,du = -\ln|u| + C$$
$$= -\ln|\cos x| + C = \ln|\sec x| + C,$$

where C is an arbitrary constant.

Remark 5.40 The solution to the previous examples may also be presented without introducing a new symbol u for the substitution. For instance, Example 5.39 can be done as follows:

$$\begin{aligned}\int \tan x\,dx &= \int \frac{\sin x}{\cos x}\,dx \\ &= -\int \frac{1}{\cos x}\,d\cos x \\ &= -\ln|\cos x| + C \\ &= \ln|\sec x| + C,\end{aligned}$$

where C is an arbitrary constant. If the substitution is simple, then the direct presentation here may be more convenient; if the substitution is complicated, then it is better to denote the substitution by a new symbol u as before.

Example 5.41 Evaluate

$$\int \sec x \, dx.$$

Solution:

Let $u = \sec x + \tan x$. Then $du = (\sec x \tan x + \sec^2 x)dx$, so by the substitution rule

$$\begin{aligned}\int \sec x \, dx &= \int \frac{\sec x (\sec x + \tan x)}{\sec x + \tan x} dx \\ &= \int \frac{\sec^2 x + \sec x \tan x}{\sec x + \tan x} dx \\ &= \int \frac{1}{u} du \\ &= \ln|u| + C \\ &= \ln|\sec x + \tan x| + C,\end{aligned}$$

Don't just stop at this step! We need to present the answer using the **original variable** x.

where C is an arbitrary constant.

Remark 5.42 It is often useful to remember the following **common substitutions**, although they are just the same as the corresponding formulas for antiderivatives:

- $\frac{1}{x^2} dx = -d\left(\frac{1}{x}\right)$
- $\frac{1}{\sqrt{x}} dx = 2\, d\sqrt{x}$
- $x^r dx = \frac{1}{r+1} dx^{r+1}$ for every $r \neq -1$
- $\frac{1}{x} dx = d \ln|x|$
- $\cos x \, dx = d \sin x$
- $\sin x \, dx = -d \cos x$
- $\sec^2 x \, dx = d \tan x$
- $\sec x \tan x \, dx = d \sec x$

Example 5.43 Evaluate the following antiderivatives.

(a) $\int \frac{x^3}{1 + x^4} dx$ (b) $\int \frac{e^{1/x}}{x^2} dx$ (c) $\int \frac{1}{\sqrt{x - x^2}} dx$

Solution:

(a) Let $u = 1 + x^4$. Then $du = 4x^3 dx$, so

$$\begin{aligned}\int \frac{x^3}{1 + x^4} dx &= \int \frac{1}{u}\left(\frac{1}{4} du\right) = \frac{1}{4} \int \frac{1}{u} du \\ &= \frac{1}{4} \ln|u| + C = \frac{1}{4} \ln(1 + x^4) + C,\end{aligned}$$

where C is an arbitrary constant.

(b) Let $u = \frac{1}{x}$. Then $du = -\frac{1}{x^2}dx$, so

$$\int \frac{e^{1/x}}{x^2}dx = \int e^u(-du) = -\int e^u du$$

$$= -e^u + C = -e^{\frac{1}{x}} + C,$$

where C is an arbitrary constant.

(c) Let $u = \sqrt{x}$. Then $x = u^2$ and $du = \frac{1}{2\sqrt{x}}dx$, so

$$\int \frac{1}{\sqrt{x - x^2}}dx = \int \frac{1}{\sqrt{1-x}}\left(\frac{1}{\sqrt{x}}dx\right) = \int \frac{1}{\sqrt{1-u^2}}(2\,du)$$

$$= 2\int \frac{1}{\sqrt{1-u^2}}du = 2\arcsin u + C$$

$$= 2\arcsin\sqrt{x} + C,$$

where C is an arbitrary constant.

Example 5.44 Evaluate the following antiderivatives.

(a) $\displaystyle\int \sin^3 x\,dx$

(b) $\displaystyle\int \sec^2 x \tan^2 x\,dx$

Solution:

(a)

$$\int \sin^3 x\,dx = \int \sin^2 x\,(\sin x\,dx) = \int (1 - \cos^2 x)(-d\cos x)$$

$$= \int (\cos^2 x - 1)d\cos x = \frac{1}{3}\cos^3 x - \cos x + C,$$

where C is an arbitrary constant.

(b)

$$\int \sec^2 x \tan^2 x\,dx = \int \tan^2 x\,(\sec^2 x\,dx)$$

$$= \int \tan^2 x\,d\tan x$$

$$= \frac{1}{3}\tan^3 x + C,$$

where C is an arbitrary constant.

Theorem 5.45 (Substitution rule for integrals) *Let g be a differentiable function such that g' is continuous on $[a,b]$, and let f be a function which is continuous on the range of g. Then*

$$\int_a^b f(g(x))g'(x)dx = \int_{g(a)}^{g(b)} f(u)du.$$

Proof. Let F be an antiderivative of f. Then by chain rule, $F(g(x))$ is an antiderivative of $f(g(x))g'(x)$; so

$$\int_a^b f(g(x))g'(x)dx = [F(g(x))]_a^b = F(g(b)) - F(g(a))$$

by the Newton-Leibniz formula. On the other hand, we also have

$$\int_{g(a)}^{g(b)} f(u)du = [F(u)]_{g(a)}^{g(b)} = F(g(b)) - F(g(a))$$

by the Newton-Leibniz formula again. ∎

Example 5.46 Evaluate the following integrals.

(a) $\displaystyle\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} e^{\sin x}\cos x\,dx$

(b) $\displaystyle\int_0^2 \frac{8x^3}{\sqrt{2x^2+1}}dx$

Solution:

(a) Let $u = \sin x$. Then $du = \cos x\,dx$, $u = -1$ when $x = -\frac{\pi}{2}$, and $u = 1$ when $x = \frac{\pi}{2}$.

Thus,

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} e^{\sin x}\cos x\,dx = \int_{-1}^{1} e^u\,du$$

Modify the integration interval to cope with u.

$$= [e^u]_{-1}^{1} = e - \frac{1}{e}.$$

(b) Let $u = 2x^2 + 1$. Then $= 4x\,dx$, $u = 1$ when $x = 0$, and $u = 9$ when $x = 2$. Thus,

$$\int_0^2 \frac{8x^3}{\sqrt{2x^2+1}}dx = \int_0^2 \frac{2x^2}{\sqrt{2x^2+1}}(4x\,dx)$$

Modify the integration interval to cope with u.

$$= \int_1^9 \frac{u-1}{\sqrt{u}}du = \int_1^9 \left(\sqrt{u} - \frac{1}{\sqrt{u}}\right)du$$

$$= \left[\frac{2}{3}u^{\frac{3}{2}} - 2u^{\frac{1}{2}}\right]_1^9 = 12 - \left(-\frac{4}{3}\right) = \frac{40}{3}.$$

Example 5.47 The following table shows some values of two functions f and g, whose derivatives f' and g' are continuous.

x	0	1	2	e
$f(x)$	2	-2	-1	5
$g(x)$	0	-1	-2	4
$f'(x)$	-4	1	3	2
$g'(x)$	-5	-2	4	-3

Using information from this table, evaluate the following integrals.

(a) $\displaystyle\int_1^{e^2}\left[\frac{f'(\ln x)}{x}+\frac{g'\left(\sqrt{x}\right)}{\sqrt{x}}\right]dx$

(b) $\displaystyle\int_1^{\sqrt{e}} xf(x^2)f'(x^2)dx$

Solution:

(a) Let

$$u=\ln x \qquad \text{and} \qquad v=\sqrt{x}.$$

Then we have

$$du=\frac{1}{x}dx,\quad u=0 \text{ when } x=1,\quad u=2 \text{ when } x=e^2;$$

$$dv=\frac{1}{2\sqrt{x}}dx,\quad v=1 \text{ when } x=1,\quad v=e \text{ when } x=e^2.$$

Thus,

$$\begin{aligned}\int_1^{e^2}\left[\frac{f'(\ln x)}{x}+\frac{g'\left(\sqrt{x}\right)}{\sqrt{x}}\right]dx &= \int_1^{e^2}\frac{f'(\ln x)}{x}dx+\int_1^{e^2}\frac{g'\left(\sqrt{x}\right)}{\sqrt{x}}dx\\ &=\int_0^2 f'(u)\,du+\int_1^e g'(v)(2dv)\\ &=[f(u)]_0^2+[2g(v)]_1^e\\ &=[f(2)-f(0)]+[2g(e)-2g(1)]\\ &=[(-1)-2]+[2(4)-2(-1)]\\ &=7\end{aligned}$$

(b) Let $u=f(x^2)$. Then $du=2xf'(x^2)dx$, $u=-2$ when $x=1$, and $u=5$ when $x=\sqrt{e}$. Thus,

$$\begin{aligned}\int_1^{\sqrt{e}} xf(x^2)f'(x^2)dx &= \int_{-2}^5\frac{1}{2}u\,du=\left[\frac{1}{4}u^2\right]_{-2}^5\\ &=\frac{25}{4}-1=\frac{21}{4}.\end{aligned}$$

In some integrals it is very difficult to find an antiderivative of the integrand directly, but the integral can still be computed by exploiting some **symmetry of the integrand** using the **substitution rule**.

Theorem 5.48 *Let* $a \geq 0$ *and let* f *be a continuous function on* $[-a, a]$.

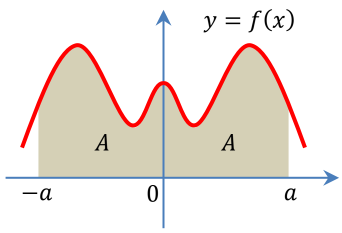
(i) *If* f *is odd, then*

$$\int_{-a}^{a} f(x)dx = 0.$$

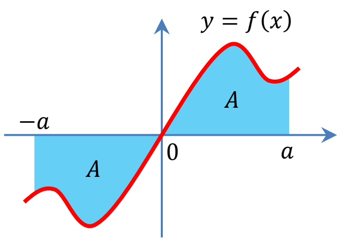
(ii) *If* f *is even, then*

$$\int_{-a}^{a} f(x)dx = 2\int_{0}^{a} f(x)dx.$$

f is **odd** if $f(-x) = -f(x)$ for every x;
f is **even** if $f(-x) = f(x)$ for every x.



$$\int_{-a}^{a} f(x)dx = A + A$$



$$\int_{-a}^{a} f(x)dx = A - A$$

Proof. We first evaluate $\int_{-a}^{0} f(x)dx$ using the substitution $u = -x$. Then we have $du = -dx$, $u = a$ when $x = -a$, and $u = 0$ when $x = 0$. Thus

$$\int_{-a}^{0} f(x)dx = -\int_{a}^{0} f(-u)du = \int_{0}^{a} f(-u)du = \int_{0}^{a} f(-x)dx.$$

(i) If f is odd, then $f(-x) = -f(x)$ for every $x \in [0, a]$, so

$$\int_{-a}^{a} f(x)dx = \int_{-a}^{0} f(x)dx + \int_{0}^{a} f(x)dx = \int_{0}^{a} f(-x)dx + \int_{0}^{a} f(x)dx$$
$$= -\int_{0}^{a} f(x)dx + \int_{0}^{a} f(x)dx = 0.$$

(ii) If f is even, then $f(-x) = f(x)$ for every $x \in [0, a]$, so

$$\int_{-a}^{a} f(x)dx = \int_{-a}^{0} f(x)dx + \int_{0}^{a} f(x)dx = \int_{0}^{a} f(-x)dx + \int_{0}^{a} f(x)dx$$
$$= \int_{0}^{a} f(x)dx + \int_{0}^{a} f(x)dx = 2\int_{0}^{a} f(x)dx.$$

■

Example 5.49 Evaluate the following integrals.

(a) $\displaystyle\int_{-1}^{1} \frac{e^x - e^{-x}}{\cos x}\,dx$

(b) $\displaystyle\int_{-1}^{1} |t|\sqrt{t^2+1}\,dt$

Solution:

(a) Let $f\colon [-1,1] \to \mathbb{R}$ be the continuous function

$$f(x) = \frac{e^x - e^{-x}}{\cos x}.$$

It is too difficult to find an antiderivative of f; but since

$$f(-x) = \frac{e^{-x} - e^{x}}{\cos(-x)} = -\frac{e^x - e^{-x}}{\cos x} = -f(x)$$

for every $x \in [-1,1]$, it follows that f is an odd function. Thus by Theorem 5.48 (i) we easily conclude that

$$\int_{-1}^{1} \frac{e^x - e^{-x}}{\cos x}\,dx = 0.$$

(b) Let $g\colon [-1,1] \to \mathbb{R}$ be the continuous function

$$g(t) = |t|\sqrt{t^2+1}.$$

It is too difficult to find an antiderivative of g; but since

$$g(-t) = |-t|\sqrt{(-t)^2+1} = |t|\sqrt{t^2+1} = g(t)$$

for every $t \in [-1,1]$, it follows that g is an even function. So by Theorem 5.48 (ii) we have

$$\begin{aligned}\int_{-1}^{1} |t|\sqrt{t^2+1}\,dt &= 2\int_{0}^{1} |t|\sqrt{t^2+1}\,dt \\ &= \int_{0}^{1} 2t\sqrt{t^2+1}\,dt\,.\end{aligned}$$

Now applying the substitution $u = t^2 + 1$, we have $du = 2t\,dt$, $u = 1$ when $t = 0$, and $u = 2$ when $t = 1$. Therefore

$$\begin{aligned}\int_{-1}^{1} |t|\sqrt{t^2+1}\,dt = \int_{0}^{1} 2t\sqrt{t^2+1}\,dt &= \int_{1}^{2} \sqrt{u}\,du \\ &= \left[\frac{2}{3}u^{\frac{3}{2}}\right]_1^2 = \frac{2}{3}\left(2\sqrt{2}-1\right).\end{aligned}$$

Example 5.50

(a) Let $f:[0,\pi]\to\mathbb{R}$ be a continuous function. Show that

$$\int_0^\pi f(x)dx=\int_0^\pi f(\pi-x)dx.$$

(b) Using the result from (a), compute the integral

$$\int_0^\pi \frac{x\sin x}{1+\cos^2 x}dx.$$

Solution:

(a) Let $u=\pi-x$. Then $du=-dx$, $u=\pi$ when $x=0$, and $u=0$ when $x=\pi$. Thus

$$\int_0^\pi f(\pi-x)dx=\int_\pi^0 f(u)(-du)=\int_0^\pi f(u)du=\int_0^\pi f(x)dx.$$

(b) Let $f:[0,\pi]\to\mathbb{R}$ be the function

$$f(x)=\frac{x\sin x}{1+\cos^2 x}$$

Then f is continuous on $[0,\pi]$. Applying the result from (a) to this function f, we have

$$\begin{aligned}\int_0^\pi \frac{x\sin x}{1+\cos^2 x}dx&=\int_0^\pi \frac{(\pi-x)\sin(\pi-x)}{1+\cos^2(\pi-x)}dx\\&=\int_0^\pi \frac{(\pi-x)\sin x}{1+\cos^2 x}dx\\&=\pi\int_0^\pi \frac{\sin x}{1+\cos^2 x}dx-\int_0^\pi \frac{x\sin x}{1+\cos^2 x}dx.\end{aligned}$$

Rearranging the above equation, we have

$$\int_0^\pi \frac{x\sin x}{1+\cos^2 x}dx=\frac{\pi}{2}\int_0^\pi \frac{\sin x}{1+\cos^2 x}dx.$$

Now applying the substitution $u=\cos x$, we have

$$\begin{aligned}\int_0^\pi \frac{\sin x}{1+\cos^2 x}dx&=\int_1^{-1}\frac{1}{1+u^2}(-du)=\int_{-1}^1\frac{1}{1+u^2}du\\&=[\arctan u]_{-1}^1=\frac{\pi}{2}.\end{aligned}$$

Therefore

$$\begin{aligned}\int_0^\pi \frac{x\sin x}{1+\cos^2 x}dx&=\frac{\pi}{2}\int_0^\pi \frac{\sin x}{1+\cos^2 x}dx\\&=\frac{\pi}{2}\cdot\frac{\pi}{2}=\frac{\pi^2}{4}.\end{aligned}$$

Remark 5.51 Antiderivatives of products are often hard to evaluate. **Trigonometric identities** are often useful in computing antiderivatives involving products of trigonometric functions.

- The **product-to-sum formulae**

$$\sin x \cos y = \frac{1}{2}[\sin(x+y) + \sin(x-y)] \qquad \cos x \sin y = \frac{1}{2}[\sin(x+y) - \sin(x-y)]$$

$$\cos x \cos y = \frac{1}{2}[\cos(x+y) + \cos(x-y)] \qquad \sin x \sin y = \frac{1}{2}[\cos(x-y) - \cos(x+y)]$$

help convert products of sines and cosines into sums, which can be handled much more easily.

- In particular, the **double angle formulae**

$$\sin^2 x = \frac{1-\cos 2x}{2} \qquad \cos^2 x = \frac{1+\cos 2x}{2} \qquad \sin x \cos x = \frac{\sin 2x}{2}$$

help reduce products of sines and cosines of the same angle.

Example 5.52 Evaluate the following integrals:

(a) $\displaystyle\int_0^{\frac{\pi}{2}} \sin 4x \cos 3x \, dx$

(b) $\displaystyle\int_0^{\frac{\pi}{2}} \cos x \cos 2x \cos 3x \, dx$

Solution:

(a) Using the product-to-sum formula, we have

$$\begin{aligned}\int_0^{\frac{\pi}{2}} \sin 4x \cos 3x \, dx &= \int_0^{\frac{\pi}{2}} \frac{1}{2}(\sin 7x + \sin x) dx \\ &= \frac{1}{2}\left[-\frac{1}{7}\cos 7x - \cos x\right]_0^{\frac{\pi}{2}} = \frac{4}{7}.\end{aligned}$$

(b) Using the product-to-sum formula, we have

$$\begin{aligned}\int_0^{\frac{\pi}{2}} \cos x \cos 2x \cos 3x \, dx &= \int_0^{\frac{\pi}{2}} \frac{1}{2}(\cos 3x + \cos x) \cos 3x \, dx \\ &= \int_0^{\frac{\pi}{2}} \left(\frac{1}{2}\cos^2 3x + \frac{1}{2}\cos x \cos 3x\right) dx \\ &= \int_0^{\frac{\pi}{2}} \left[\frac{1}{2}\left(\frac{1+\cos 6x}{2}\right) + \frac{1}{4}(\cos 4x + \cos 2x)\right] dx \\ &= \int_0^{\frac{\pi}{2}} \left(\frac{1}{4} + \frac{1}{4}\cos 2x + \frac{1}{4}\cos 4x + \frac{1}{4}\cos 6x\right) dx \\ &= \left[\frac{1}{4}x + \frac{1}{8}\sin 2x + \frac{1}{16}\sin 4x + \frac{1}{24}\sin 6x\right]_0^{\frac{\pi}{2}} = \frac{\pi}{8}.\end{aligned}$$

When we encounter antiderivatives or integrals of **trigonometric functions**, we often need to decide between using the **substitution rule** or using **trigonometric identities**.

Example 5.53 Evaluate the following antiderivatives:

(a) $\int \cos^2 x\,dx$

(b) $\int \cos^3 x\,dx$

(c) $\int \cos^4 x\,dx$

(d) $\int \cos^5 x\,dx$

Solution:

(a) Using the double angle formula, we have

$$\int \cos^2 x\,dx = \int \frac{1+\cos 2x}{2}dx = \frac{1}{2}\int dx + \frac{1}{2}\int \cos 2x\,dx = \frac{1}{2}x + \frac{1}{4}\sin 2x + C,$$

where C is an arbitrary constant.

(b) Using the substitution $u = \sin x$, we have $du = \cos x\,dx$. Thus

$$\int \cos^3 x\,dx = \int \cos^2 x\,(\cos x\,dx) = \int (1-\sin^2 x)d\sin x = \sin x - \frac{1}{3}\sin^3 x + C,$$

where C is an arbitrary constant.

(c) Using the double angle formula twice, we have

$$\begin{aligned}\int \cos^4 x\,dx &= \int (\cos^2 x)^2 dx = \int \left(\frac{1+\cos 2x}{2}\right)^2 dx = \int \frac{1}{4}(1+2\cos 2x+\cos^2 2x)dx \\ &= \int \frac{1}{4}\left(1+2\cos 2x+\frac{1+\cos 4x}{2}\right)dx \\ &= \frac{3}{8}\int dx + \frac{1}{2}\int \cos 2x\,dx + \frac{1}{8}\int \cos 4x\,dx \\ &= \frac{3}{8}x + \frac{1}{4}\sin 2x + \frac{1}{32}\sin 4x + C,\end{aligned}$$

where C is an arbitrary constant.

(d) Using the substitution $u = \sin x$, we have $du = \cos x\,dx$. Thus

$$\begin{aligned}\int \cos^5 x\,dx &= \int \cos^4 x\,(\cos x\,dx) = \int (1-\sin^2 x)^2\,d\sin x \\ &= \int (1-2\sin^2 x+\sin^4 x)\,d\sin x = \sin x - \frac{2}{3}\sin^3 x + \frac{1}{5}\sin^5 x + C,\end{aligned}$$

where C is an arbitrary constant.

Think: Why shouldn't we use the substitution $\cos x\,dx = d\sin x$ in (a) or (c)? What happens if we use double angle formula in (b) or (d)?

Example 5.54 Evaluate the following antiderivatives:

(a) $\int \sin^2 x \cos^3 x \, dx$

(b) $\int \sin^3 x \cos^4 x \, dx$

(c) $\int \sin^2 x \cos^4 x \, dx$

Solution:

(a) Using the substitution $u = \sin x$, we have $du = \cos x \, dx$. Thus

$$\int \sin^2 x \cos^3 x \, dx = \int \sin^2 x \cos^2 x \, (\cos x \, dx) = \int \sin^2 x \, (1 - \sin^2 x) \, d \sin x$$

$$= \int (\sin^2 x - \sin^4 x) \, d \sin x = \frac{1}{3} \sin^3 x - \frac{1}{5} \sin^5 x + C,$$

where C is an arbitrary constant.

(b) Using the substitution $u = \cos x$, we have $du = -\sin x \, dx$. Thus

$$\int \sin^3 x \cos^4 x \, dx = \int -\sin^2 x \cos^4 x \, (-\sin x \, dx) = \int (\cos^2 x - 1) \cos^4 x \, d \cos x$$

$$= \int (\cos^6 x - \cos^4 x) \, d \cos x = \frac{1}{7} \cos^7 x - \frac{1}{5} \cos^5 x + C,$$

where C is an arbitrary constant.

(c) Using the double angle formula and product-to-sum formula, we have

$$\int \sin^2 x \cos^4 x \, dx = \int \frac{1 - \cos 2x}{2} \left(\frac{1 + \cos 2x}{2} \right)^2 dx = \frac{1}{8} \int (1 - \cos^2 2x)(1 + \cos 2x) dx$$

$$= \frac{1}{8} \int \left(1 - \frac{1 + \cos 4x}{2} \right) (1 + \cos 2x) dx$$

$$= \frac{1}{16} \int (1 + \cos 2x - \cos 4x - \cos 4x \cos 2x) dx$$

$$= \frac{1}{16} \int \left[1 + \cos 2x - \cos 4x - \frac{1}{2} (\cos 6x + \cos 2x) \right] dx$$

$$= \frac{1}{16} \int dx + \frac{1}{32} \int \cos 2x \, dx - \frac{1}{16} \int \cos 4x \, dx - \frac{1}{32} \int \cos 6x \, dx$$

$$= \frac{1}{16} x + \frac{1}{64} \sin 2x - \frac{1}{64} \sin 4x - \frac{1}{192} \sin 6x + C,$$

where C is an arbitrary constant.

Examples 5.53 and 5.54 suggest that $\int \sin^m x \cos^n x \, dx$ can be evaluated easily by substitution if $\boldsymbol{m}$ **is odd or** $\boldsymbol{n}$ **is odd**. In case both m and n are even, it would be better to use trigonometric identities or some other method which we will study in the next section.

4. Integration by parts

The following rule of antidifferentiation corresponds to the **product rule** of differentiation.

Theorem 5.55 (Antidifferentiation by parts) *Let f and g be differentiable functions. Then*

$$\int f(x)g'(x)dx = f(x)g(x) - \int f'(x)g(x)dx.$$

If we let $u = f(x)$ and $v = g(x)$, then $du = f'(x)dx$ and $dv = g'(x)dx$, so the above can also be written as

$$\int u\,dv = uv - \int v\,du.$$

Proof. The derivative of the product fg is given by $(fg)' = f'g + fg'$, so fg is an antiderivative of $f'g + fg'$. In other words, we have

$$\int [f'(x)g(x) + f(x)g'(x)]dx = f(x)g(x) + C,$$

where C is an arbitrary constant, i.e.

$$\int f(x)g'(x)dx = f(x)g(x) - \int f'(x)g(x)dx.$$

■

Corollary 5.56 (Integration by parts) *Let f and g be differentiable functions such that f' and g' are continuous on $[a, b]$. Then*

$$\int_a^b f(x)g'(x)dx = [f(x)g(x)]_a^b - \int_a^b f'(x)g(x)dx.$$

Example 5.57 Evaluate

$$\int xe^x dx.$$

Solution:
We let $f(x) = x$ and $g'(x) = e^x$ (so that we may take $g(x) = e^x$). Then

$$\begin{aligned}\int \underbrace{x}_{f(x)}\,\underbrace{e^x}_{g'(x)}\,dx &= \underbrace{x}_{f(x)}\,\underbrace{e^x}_{g(x)} - \int \underbrace{1}_{f'(x)} \cdot \underbrace{e^x}_{g(x)}\,dx \\ &= xe^x - e^x + C,\end{aligned}$$

where C is an arbitrary constant.

Example 5.58 Evaluate the following antiderivatives.

(a) $\displaystyle\int \ln x\,dx$ (b) $\displaystyle\int \arctan x\,dx$

Solution:

(a) We let $u = \ln x$ and $dv = dx$ (so that we may take $v = x$). Then

$$\int \underbrace{\ln x}_{u}\,\underbrace{dx}_{dv} = \underbrace{(\ln x)}_{u}\,\underbrace{(x)}_{v} - \int \underbrace{x}_{v} \cdot \underbrace{\frac{1}{x}dx}_{du}$$

$$= x\ln x - \int dx$$

$$= x\ln x - x + C,$$

where C is an arbitrary constant.

(b) In a similar way, we let $u = \arctan x$ and $dv = dx$ (so that we may take $v = x$). Then

$$\int \underbrace{\arctan x}_{u}\,\underbrace{dx}_{dv} = \underbrace{(\arctan x)}_{u}\,\underbrace{(x)}_{v} - \int \underbrace{x}_{v} \cdot \underbrace{\frac{1}{1+x^2}dx}_{du}$$

$$= x\arctan x - \frac{1}{2}\int \frac{1}{1+x^2}\,d(1+x^2)$$

$$= x\arctan x - \frac{1}{2}\ln(1+x^2) + C,$$

where C is an arbitrary constant.

Example 5.59 Evaluate the integral

$$\int_1^2 \frac{\ln x}{x^2}\,dx.$$

Solution:

We let $u = \ln x$ and $dv = \frac{1}{x^2}dx$ (so that we may take $v = -\frac{1}{x}$). Integrating by parts, we get

$$\int_1^2 \frac{\ln x}{x^2}\,dx = \left[(\ln x)\left(-\frac{1}{x}\right)\right]_1^2 - \int_1^2 -\frac{1}{x}\cdot\frac{1}{x}\,dx$$

$$= \left[-\frac{1}{x}\ln x\right]_1^2 - \int_1^2 -\frac{1}{x^2}\,dx$$

$$= \left[-\frac{1}{x}\ln x\right]_1^2 - \left[\frac{1}{x}\right]_1^2 = \frac{1}{2} - \frac{1}{2}\ln 2\,.$$

Example 5.60 Evaluate

$$\int_0^{\pi} x^2 \cos x \, dx.$$

Solution:
Let $f(x) = x^2$ and $g'(x) = \cos x$ (so that we may take $g(x) = \sin x$). Then integrating by parts we get

$$\begin{aligned}\int_0^{\pi} x^2 \cos x \, dx &= [x^2 \sin x]_0^{\pi} - \int_0^{\pi} 2x \sin x \, dx \\ &= (\pi^2 \sin \pi - 0^2 \sin 0) - \int_0^{\pi} 2x \sin x \, dx \\ &= \int_0^{\pi} -2x \sin x \, dx.\end{aligned}$$

To deal with the last integral, we use integration by parts again. This time we let $f(x) = 2x$ and $g'(x) = -\sin x$ (so that we may take $g(x) = \cos x$). Then

$$\begin{aligned}\int_0^{\pi} x^2 \cos x \, dx = \int_0^{\pi} -2x \sin x \, dx &= [2x \cos x]_0^{\pi} - \int_0^{\pi} 2 \cos x \, dx \\ &= [2x \cos x - 2 \sin x]_0^{\pi} = -2\pi.\end{aligned}$$

Remark 5.61 When we need to do integration by parts multiple times, there is an efficient **tabular method** for us to come up with the correct final formula. Suppose that we want to compute the antiderivatives $\int fg \, dx$ by parts for four times, say. We then construct the following table.

sign	**D**	**I**
$+$	f	g
$-$	f'	$\int g$
$+$	f''	$\int \int g$
$-$	f'''	$\int \int \int g$
$+$	$f^{(4)}$	$\int \int \int \int g$

We then multiply along the red arrows and sum up the terms, followed by adding the integral of the product of the last row (blue arrow).

sign	**D**	**I**
$+$	f	g
$-$	f'	$\int g$
$+$	f''	$\int \int g$
$-$	f'''	$\int \int \int g$
$+$	$f^{(4)}$	$\int \int \int \int g$

This gives us the correct formula for integration by parts.

Example 5.62 Evaluate

$$\int x^4 \sin x \, dx.$$

Solution:

Let $f(x) = x^4$ and $g(x) = \sin x$. Then we construct the following table:

sign	D	I
$+$	x^4	$\sin x$
$-$	$4x^3$	$-\cos x$
$+$	$12x^2$	$-\sin x$
$-$	$24x$	$\cos x$
$+$	24	$\sin x$

from which we get

$$\int x^4 \sin x \, dx = -x^4 \cos x + 4x^3 \sin x + 12x^2 \cos x - 24x \sin x + \int 24 \sin x \, dx$$

$$= -x^4 \cos x + 4x^3 \sin x + 12x^2 \cos x - 24x \sin x - 24 \cos x + C,$$

where C is an arbitrary constant.

Example 5.63 Evaluate

$$\int_0^\pi e^x \cos x \, dx\,.$$

Solution:

Let $f(x) = e^x$ and $g(x) = \cos x$. Then we construct the following table:

sign	D	I
$+$	e^x	$\cos x$
$-$	e^x	$\sin x$
$+$	e^x	$-\cos x$

from which we get

$$\int_0^\pi e^x \cos x \, dx = [e^x \sin x + e^x \cos x]_0^\pi - \int_0^\pi e^x \cos x \, dx = -e^\pi - 1 - \int_0^\pi e^x \cos x \, dx.$$

Now the required integral $\int_0^\pi e^x \cos x \, dx$ appears on both sides of the equation, so rearranging the terms we get $2\int_0^\pi e^x \cos x \, dx = -e^\pi - 1$, i.e.

$$\int_0^\pi e^x \cos x \, dx = -\frac{e^\pi + 1}{2}.$$

Sometimes we need to apply certain substitution(s) before we use integration by parts.

Example 5.64 Evaluate the following antiderivatives.

(a) $\displaystyle\int e^x \arctan(e^x)\,dx$

(b) $\displaystyle\int \sqrt{x}e^{\sqrt{x}}dx$

Solution:

(a) Let $u = e^x$. Then $du = e^x dx$, so

$$\int e^x \arctan(e^x)\,dx = \int \arctan u\,du\,.$$

In Example 5.58 (b) we have already seen that

$$\int \arctan u\,du = u\arctan u - \frac{1}{2}\ln(1+u^2) + C,$$

where C is an arbitrary constant. Replacing u be e^x again, we obtain

$$\int e^x \arctan(e^x)\,dx = e^x \arctan(e^x) - \frac{1}{2}\ln(1+e^{2x}) + C,$$

where C is an arbitrary constant.

(b) Let $u = \sqrt{x}$. Then $x = u^2$ and $dx = 2u\,du$, so

$$\int \sqrt{x}e^{\sqrt{x}}dx = \int ue^u(2u\,du) = \int 2u^2e^u du\,.$$

Now we apply antidifferentiation by parts twice:

sign	D	I
$+$	$2u^2$	e^u
$-$	$4u$	e^u
$+$	4	e^u

We obtain

$$\int 2u^2e^u du = 2u^2e^u - 4ue^u + \int 4e^u du = 2u^2e^u - 4ue^u + 4e^u + C,$$

where C is an arbitrary constant. Replacing u by $\sqrt{x}$ again, we obtain

$$\int \sqrt{x}e^{\sqrt{x}}dx = 2xe^{\sqrt{x}} - 4\sqrt{x}e^{\sqrt{x}} + 4e^{\sqrt{x}} + C,$$

where C is an arbitrary constant.

5. Net change

In this final section, let's apply the concepts of antidifferentiation and integration in solving some simple problems in geometry, in kinematics, or in daily life.

Remark 5.65 Using antidifferentiation, we can solve a **differential equation** of the form

$$\frac{dy}{dx} = f(x).$$

To "solve" such a differential equation means to find a possible candidate for the **unknown function** y; any antiderivative of f would do the job.

Example 5.66 Solve the following differential equations:

(a) $\dfrac{dy}{d\theta} = \dfrac{\sin\theta}{\cos^2\theta}$

(b) $\dfrac{du}{dx} = 2^{x-2}$

Solution:

(a) Since $\frac{dy}{d\theta} = \frac{\sin\theta}{\cos^2\theta} = \sec\theta\tan\theta$, taking antiderivatives on both sides, we get

$$y = \int \sec\theta\tan\theta\, d\theta = \sec\theta + C,$$

where C is an arbitrary constant.

(b) Since $\frac{du}{dx} = 2^{x-2} = \frac{1}{4}2^x$, taking antiderivatives on both sides, we get

$$u = \int \frac{1}{4}2^x dx = \frac{1}{4}\int 2^x dx = \frac{1}{4}\frac{2^x}{\ln 2} + C = \frac{1}{\ln 2}2^{x-2} + C,$$

where C is an arbitrary constant.

Remark 5.67 A general solution to the differential equation

$$\frac{dy}{dx} = f(x)$$

must involve an arbitrary constant. However, if we further require that the solution function y must take a certain chosen value at a number a, e.g. $y(a) = b$, then the value of the arbitrary constant can be determined, so that on each open interval containing a there is **only one** solution function satisfying this additional requirement. This additional condition $y(a) = b$ imposed on the solution is called an ***initial condition***. A differential equation together with an initial condition is called an ***initial value problem***.

Example 5.68 Suppose that at each point (x, y) on a curve, the slope of the tangent line to the curve is $x + \frac{1}{x^2}$. If the curve cuts the x-axis at $x = 2$, find the equation of the curve.

Solution:

The slope of the tangent to a curve is given by the derivative of the function it represents, i.e.

$$\frac{dy}{dx} = x + \frac{1}{x^2}.$$

This problem is equivalent to the initial value problem $\begin{cases} \frac{dy}{dx} = x + \frac{1}{x^2}. \\ y(2) = 0 \end{cases}$

Taking antiderivatives on both sides, we have the equation

$$y = \frac{1}{2}x^2 - \frac{1}{x} + C,$$

where C is some constant. Since the curve passes through the point $(2, 0)$, we have

$$0 = \frac{1}{2}(2^2) - \frac{1}{2} + C,$$

so $C = -\frac{3}{2}$. Therefore the equation of the curve is

$$y = \frac{1}{2}x^2 - \frac{1}{x} - \frac{3}{2}.$$

Example 5.69 It is given that at each point (x, y) on a certain curve, we have

$$\frac{d^2y}{dx^2} = \sin x + e^x.$$

If the curve passes through the point $(0, -1)$ and the slope of the tangent line to the curve at this point is 3, find the equation of the curve.

Solution:

Take antiderivatives twice on both sides of the given equation $\frac{d^2y}{dx^2} = \sin x + e^x$, we get

$$\frac{dy}{dx} = -\cos x + e^x + C$$

and

$$y = -\sin x + e^x + Cx + D,$$

This problem is equivalent to the initial value problem $\begin{cases} y''(x) = \sin x + e^x \\ y(0) = -1 \\ y'(0) = 3 \end{cases}.$

where C and D are some constants.

Since the curve passes through the point $(0, -1)$, at which the slope of the tangent is 3, we have

$$\begin{cases} -\cos 0 + e^0 + C = 3 \\ -\sin 0 + e^0 + C(0) + D = -1 \end{cases}.$$

On solving this system we get $C = 3$ and $D = -2$, so the equation of the curve is

$$y = -\sin x + e^x + 3x - 2.$$

Remark 5.70 Recall that the position $x(t)$, the velocity $v(t)$ and the acceleration $a(t)$ of a moving particle at time t satisfy the relations $a = v'$ and $v = x'$. Therefore

$$\int a(t)dt = v(t) + C \qquad \text{and} \qquad \int v(t)dt = x(t) + D,$$

for some constants C and D.

Example 5.71 A man pushes a trolley so that it travels down a slope with some initial speed $u\ \mathrm{m/s}$. The trolley accelerates at a constant rate of $2\ \mathrm{m/s^2}$, and arrives at the foot of the slope $130\ \mathrm{m}$ away from the starting point 10 seconds later. Find the value of u.

Solution:
Let $x(t)$ and $v(t)$ be respectively the position and the velocity of the trolley t seconds after the man pushes it. Then

$$v(t) = \int a(t)dt = \int 2\,dt = 2t + C$$

and

$$x(t) = \int v(t)dt = \int (2t + C)dt = t^2 + Ct + D,$$

where C and D are some constants.

Now we have $x(0) = 0$ and $x(10) = 130$, so we obtain the system

$$\begin{cases} 0^2 + C(0) + D = 0 \\ 10^2 + C(10) + D = 130 \end{cases},$$

which gives $D = 0$ and $C = 3$.
Therefore the initial speed u is given by

$$u = v(0) = 2(0) + 3 = 3.$$

Recall that in Definition 4.1, we have defined the **rate of change** of a physical quantity (with respect to an independent variable, e.g. time) to be its **derivative**. Now thanks to the Newton-Leibniz formula again, we can also apply **integrals** in solving problems that involve various kinds of **physical quantities**, although the idea of integrals originated from a **geometry problem** of finding areas.

Corollary 5.72 (Net change) *Let f be a differentiable function whose derivative f' is continuous, and let $y = f(x)$ be a physical quantity which depends on the variable x. Then for every $a < b$, the* ***net change*** *of y as x changes from a to b is given by the* ***integral of the rate of change*** *f' over the interval $[a, b]$, i.e.*

$$f(b) - f(a) = \int_a^b f'(x)dx.$$

Example 5.73 Here are some examples of interpreting Corollary 5.72 in real-life contexts:

- If $V'(t)$ is the **rate of water flowing** into a reservoir at time t, then
$$\int_a^b V'(t)dt = V(b) - V(a)$$
is the net change of volume of water in the reservoir from time a to time b.
- If $n'(t)$ is the **rate of change of the population** of a city at time t, then
$$\int_a^b n'(t)dt = n(b) - n(a)$$
is the net change in population of the city from time a to time b.
- If $C'(x)$ is the **marginal cost** of producing one more unit of goods when the production is at x units, then
$$\int_a^b C'(x)dx = C(b) - C(a)$$
is the increase in the total cost when the production is increased from a units to b units.

Example 5.74 If oil leaks from a tank at a constant rate of $r(t) = 50$ liters per minute, then in two hours the total amount of oil leaked out (in liters) is
$$\int_0^{2\times 60} r(t)dt = \int_0^{120} 50\,dt = [50t]_0^{120} = 6000.$$

Example 5.75 At time $t = 0$ minute, water starts flowing into an empty tank of capacity $48\ \mathrm{m}^3$ at a varying rate of
$$r(t) = 9\sqrt{t}$$
(in $\mathrm{m}^3/\mathrm{min}$). How long does it take to fill up the whole tank completely?

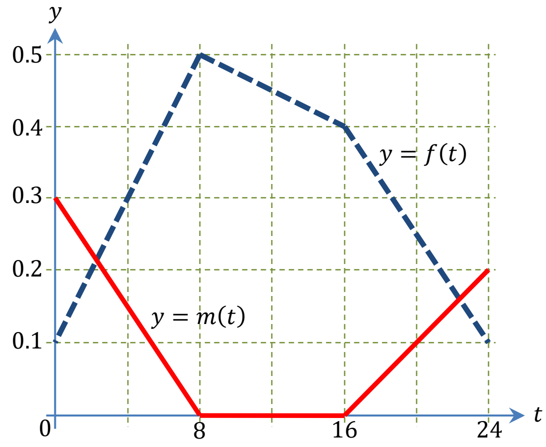
Solution:

Let $V(s)$ be the total volume of water (in m^3) flowed into the tank after s minutes. Then we have $V(0) = 0$ since the tank is empty at the beginning. The rate of change of V is given by $r(t) = 9\sqrt{t}$, so by the net change theorem we also have
$$V(s) - V(0) = \int_0^s r(t)dt = \int_0^s 9\sqrt{t}\,dt = \left[6t^{\frac{3}{2}}\right]_0^s = 6s^{\frac{3}{2}},$$
i.e.
$$V(s) = 6s^{\frac{3}{2}}.$$
Filling up the whole tank completely means that $V(s) = 48$, i.e. $s = 4$. Therefore it takes 4 minutes to fill up the whole tank completely.

Example 5.76 Let $f, m: [0, 24] \to \mathbb{R}$ be functions whose graphs are as shown in the diagram on the right.

- $f(t)$ (dashed graph, inches per hour) represents the **rate of snowfall** and
- $m(t)$ (solid graph, inches per hour) represents the **rate of snow melt**

at a location at time t (in hours). If there was 15 inches of snow on the ground at $t = 0$ hours, how much snow was there on the ground at $t = 24$ hours?



Solution:

Let $h(t)$ be the depth of the snow on the ground (in inches) at time t. Then we have $h(0) = 15$, and the issue is to calculate $h(24)$. Now the rate of change of the depth of snow is

$$h'(t) = f(t) - m(t),$$

so

$$\begin{aligned} h(24) - h(0) &= \int_0^{24} h'(t)dt = \int_0^{24} f(t)dt - \int_0^{24} m(t)dt \\ &= (\text{Area under the graph of } f) - (\text{Area under the graph of } m) \\ &= 8 - 2 = 6. \end{aligned}$$

Therefore

$$h(24) = h(0) + 6 = 15 + 6 = 21,$$

i.e. there was 21 inches of snow on the ground at $t = 24$ hours.

Remark 5.77 Let $x(t)$, $v(t)$ and $A(t)$ denote respectively the position, the velocity, and the acceleration at time t of a particle moving along a straight line.

- Since $v = x'$ and $A = v'$, Corollary 5.72 says that the **net change of the velocity** and the **displacement** (i.e. **net change of the position**) of the particle from a time a to a time b are respectively given by

$$v(b) - v(a) = \int_a^b A(t)dt \qquad \text{and} \qquad x(b) - x(a) = \int_a^b v(t)dt.$$

- The **speed** of the particle at time t is given by $|v(t)|$, i.e. the absolute value of its velocity. The **distance** traveled from a time a to a time b is the integral of the speed function:

$$\int_a^b |v(t)|dt.$$

Example 5.78 A particle moves along a line with velocity function

$$v(t) = t^2 - 2t$$

meters per second after t seconds. Find

(a) the displacement of this particle from its initial position, and

(b) the distance it has traveled

after 6 seconds.

Solution:

(a) The displacement of this particle (in meters) from its initial position after 6 seconds is

$$x(6) - x(0) = \int_0^6 (t^2 - 2t)dt = \left[\frac{1}{3}t^3 - t^2\right]_0^6 = 36.$$

(b) The distance (in meters) it has traveled after 6 seconds is

$$\begin{aligned} d = \int_0^6 |t^2 - 2t|dt &= \int_0^2 -(t^2 - 2t)dt + \int_2^6 (t^2 - 2t)dt \\ &= \left[-\frac{1}{3}t^3 + t^2\right]_0^2 + \left[\frac{1}{3}t^3 - t^2\right]_2^6 = \frac{116}{3}. \end{aligned}$$

Example 5.79 Suppose that a particle is moving along a straight line such that its velocity is

$$v(t) = 5\cos\frac{\pi t}{3}$$

meters per second after t seconds. Find the displacement of this particle from its initial position after T seconds. Also determine the time(s) when the particle returns to its initial position.

Solution:

The displacement of this particle (in meters) from its initial position after T seconds is

$$\begin{aligned} x(T) - x(0) = \int_0^T v(t)dt &= \int_0^T 5\cos\frac{\pi t}{3}dt \\ &= 5\left[\frac{3}{\pi}\sin\frac{\pi t}{3}\right]_0^T = \frac{15}{\pi}\sin\frac{\pi T}{3}. \end{aligned}$$

Such a displacement is zero when $\frac{\pi T}{3} = n\pi$, i.e. when

$$T = 3n,$$

where n is an integer. So this particle returns to its initial position after every other 3 seconds.

In the next course MATH1014, we will study integrals from a conceptual point of view. We will also study more about various techniques and applications of integration to a greater depth.

Summary of Chapter 5

The following are what you need to know in this chapter in order to get a pass (a distinction) in this course:

- ✓ To interpret an **integral** as the (signed) area of a region under a graph
- ✓ To **evaluate integrals** using known results from geometry
- ✓ **Properties** of integrals
  - ⦿ Equality: **Area property**
  - ⦿ Inequality: **Integral inequality**

- ✓ **Newton-Leibniz formula** $\int_a^b f(x)dx = F(b) - F(a)$
- ✓ **Antidifferentiation** as a reverse process of differentiation
- ✓ **Arbitrary constant** in an antiderivative
- ✓ **Rules** of antidifferentiation and integration
  - ⦿ Sum rule, difference rule, constant multiple rule
  - ⦿ **Substitution rule**
  - ⦿ **Integration by parts**
- ✓ **Antidifferentiation formulas**, to evaluate **integrals** using antiderivatives

- ✓ **Techniques** of antidifferentiation and integration
  - ⦿ To evaluate integrals of **functions involving absolute values**
  - ⦿ To evaluate antiderivatives and integrals **by substitution**
  - ⦿ To evaluate antiderivatives and integrals **by parts**
  - ⦿ Using trigonometric identities, e.g. **double angle formula** or **product-to-sum formula**
  - ⦿ To evaluate integrals of **odd**, **even** and **periodic functions** over appropriate intervals

- ✓ **Differential equations** of the form $\frac{dy}{dx} = f(x)$, **initial value problems**
- ✓ To interpret integrals as **physical quantities**
- ✓ To apply integration in computing the **net change** of a physical quantity given its **rate of change**